Pulse Dynamics in an Excitable Reaction – Diffusion System

T. Ohta [†], T. Ito [#] *

[†] Institute for Nonlinear Sciences and Applied Mathematics, Graduate School of Sciences, Hiroshima University, Higashi-Hiroshima, 739-8526 Japan

[#] Graduate School of Humanities and Sciences, Ochanomizu University, Tokyo, 112-0012 Japan

SUMMARY: We formulate a theory for pulse dynamics in an excitable reaction-diffusion system not only in one dimension but also in higher dimensions. In the singular limit where the width of pulse boundaries is infinitesimally thin we derive the equation of motion for a pair of interacting pulses. This equation contains the bifurcation that a motionless pulse loses stability and begins to propagate. The theory predicts the following remarkable properties.When the system is far away from the bifurcation threshold, two propagating pulses merge each other upon head-on collision. However when the system is close to the threshold the pulses behave as if they are elastic objects. These results are consistent with recent computer simulations.

Introduction

Recent computer simulations of reaction-diffusion equations have uncovered a number of curious behaviors of pulses in nonlinear dissipative systems. First of all, stable pulses upon head-on collision have been found [1-4], which are apparently similar to solitons in integrable systems. However, one of the essential differences from solitons is that preservation of pulses is observed only in a restricted parameter regime. If the system is bistable in a sense that a linearly stable uniform solution and stable limit cycle oscillation coexist, collision of propagating pulses causes a localized oscillatory domain which emits persistently a pair of outgoing wavetrains [5]. It should also be mentioned that, in this Moscow conference, Mornev and coworkers have reported almost the same results [6, 7] which they found independently of us.

Another interesting phenomenon is a self-replication of pulse. This has been obtained in several reaction-diffusion systems not only in one dimension but also in two dimensions [8, 9] (and in a real experiment of chemical reaction [10]). We have shown that an interplay of

 CCC 1022-1360/00/$ 17.50+.50/0

preservation, pair annihilation and self-replication produces a regular self-similar spatio-temporal pattern equivalent with a Shierpinsky gasket [11, 12].

In this paper, we focus our attention on the preservation (or more specifically reflection) of propagating pulses upon collision and formulate a theory for pulse dynamics in an excitable reaction diffusion system.

Excitable reaction Diffusion System

We start with a coupled set of reaction-diffusion equations for activator u and inhibitor v.

$$\tau\varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 \nabla^2 u + f\{u, v\} - v \tag{1}$$

$$\frac{\partial v}{\partial t} = D\nabla^2 v + u - \gamma v \tag{2}$$

where $0 < \varepsilon << 1$ and

$$f\{u, v\} = -u + \theta(u - a'\{u, v\}) \tag{3}$$

with $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Note that there is a global coupling through a' given by

$$a' = a + \sigma\left[\int (u + v) d\vec{r} - S_0\right] \tag{4}$$

where σ and S_0 are positive constants, $0 < a < 1/2$ and integral runs over the whole space. Equation (4) was introduced by Krisher and Mikhailov [3]. The constants τ and γ are positive and chosen such that the system is excitable and that a localized stable pulse solution exists.

It has been known that a motionless pulse is stable when τ is sufficiently large. There is a bifurcation such that below threshold value τ_B motionless pulse loses stability and undergoes breathing motion. In order to study a collision of propagating pulse pairs, one must avoid the breathing instability. For this purpose, we take the limit $\sigma \to \infty$ so that (4) becomes $a' = a$ with $\int d\vec{r}(u + v) = S_0$. This means that the volume of excited domain is independent of time.

Pulse equation of motion

Suppose that there are a pair of interacting domains. Hereafter we will call an excited domain a pulse not only in one dimension but also in higher dimensions. We will derive the equation of motion for the center of gravity $\vec{\rho}_n$ of n-th pulse (n = 1, 2) by means of the singular perturbation. We do not describe the details of the derivation but show the essential steps of the theory.

First of all, we consider a motion of an isolated spherical pulse with a radius R. As mentioned above, the radius R is constant and is related with S_0. In the limit $\varepsilon \to 0$, one obtains from (1)

$$-u + \theta(u - a) - v = 0 \tag{5}$$

Defining the location of the pulse boundary by $u = a$, one has from (5)

$$u = \theta(R - |\vec{r} - \vec{\rho}|) - v \tag{6}$$

Substituting this into (2) yields

$$\frac{\partial v}{\partial t} = D\nabla^2 v + \theta(R - |\vec{r} - \vec{\rho}|) - \beta v \tag{7}$$

with $\beta = 1 + \gamma$. We deal with $\partial v/\partial t$ of (7) as a perturbation assuming that the pulse velocity is sufficiently small and v relaxes rapidly for a given u. As a result, one has an expansion

$$v = v_0 + v_1 \tag{8}$$

$$v_0 = G\theta \tag{9}$$

$$v_1 = -G^2 \frac{\partial \theta}{\partial t} + G^3 \frac{\partial^2 \theta}{\partial t^2} - G^4 \frac{\partial^3 \theta}{\partial t^3} + \ldots \tag{10}$$

where G is defined through the relation

$$\left(-D\nabla^2 + \beta\right) G(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \tag{11}$$

and $GA = \int d\vec{r}\,' G(\vec{r} - \vec{r}\,') A(\vec{r}\,')$.

Putting $u = \bar{u}(\vec{r} - \vec{\rho})$, Eq. (1) can be written as

$$-\tau\varepsilon\dot{\rho}^i \nabla^i \bar{u} = \varepsilon^2 \nabla^2 \bar{u} - \bar{u} + \theta(\bar{u} - a) - v \tag{12}$$

where ρ^i is the i-th component of $\vec{\rho}$ and a dot means a time derivative.The repeated indices imply summation. Note here that since we are concerned with the length scale $\varepsilon << 1$, we may regard v in the last term as a constant. The solution of (12) can be written as

$$\bar{u} = \bar{u}_0 + \bar{u}_1 \tag{13}$$

$$\nabla^i \bar{u}_0 = M \nabla^i \theta \tag{14}$$

$$\bar{u}_1 = \tau\varepsilon\dot{\rho}^i M\nabla^i\bar{u}_0 + (\tau\varepsilon)^2\dot{\rho}^i\dot{\rho}^j M^2\nabla^i\nabla^j\bar{u}_0 + \dots \tag{15}$$

where $\bar{u}_0$ is the equilibrium solution and $\bar{u}_1$ is the correction due to $\dot{\vec{\rho}} \neq 0$. The right hand side of (14) has the same meaning as in (10) where the operator M is defined through

$$\left(-\varepsilon^2\nabla^2 + 1\right)M(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \tag{16}$$

Strictly speaking, there is another term of the form of $-\delta(u_0 - a)M(\vec{r} - \vec{r}\,')$. However, this is incorporated into a boundary condition and is omitted in (16).

Substituting $u = \bar{u}(\vec{r} - \vec{\rho}) + \delta u$ into (1), one obtains from the solvability condition

$$\begin{aligned}&\ddot{\rho}^j\left(\nabla^i\theta, G^3\nabla^j\theta\right) + \left[\tau\varepsilon\left(\nabla^i\bar{u}_0, \nabla^j\bar{u}_0\right) - \left(\nabla^i\theta, G^2\nabla^j\theta\right)\right]\dot{\rho}^j + \\ &+ \left[\left(\nabla^i\nabla^l\theta, G^4\nabla^j\nabla^k\theta\right) - (\tau\varepsilon)^3\left(\nabla^i\nabla^l\bar{u}_0, M^2\nabla^j\nabla^k\bar{u}_0\right)\right]\dot{\rho}^j\dot{\rho}^k\dot{\rho}^l = 0\end{aligned} \tag{17}$$

where $(A, B) = \int d\vec{r} A(\vec{r})B(\vec{r})$. We have ignored the higher order time derivatives and used the relations such as $\partial\theta/\partial t = -\dot{\vec{\rho}} \cdot \vec{\nabla}\theta$ in (10).

Now we have to take account of the interaction between two pulses having the same radius. Since the interaction is a static property we derive it for a motionless pulse pair, which is valid up to the leading order of the pulse velocity. When $|\vec{\rho}_1 - \vec{\rho}_2| >> R$, one may approximate as

$$u = \bar{u}_0^{(1)} + \bar{u}_0^{(2)} \tag{18}$$

where $\bar{u}_0^{(n)} = \bar{u}_0(\vec{r} - \vec{\rho}_n)$, (n = 1, 2). Since the absolute value of $u_0^{(2)}$ is quite small in the vicinity of the first pulse, the effect of the second pulse can be treated as a perturbation. Using the fact that $\bar{u}_0^{(n)}$ is the equilibrium solution of (12), one obtains

$$\tau\varepsilon\left(\nabla^i\bar{u}_0^{(1)}, \nabla^j\bar{u}_0^{(1)}\right)\dot{\rho}_1^j = C_d\nabla_1^i\bar{u}_0(|\vec{\rho}_1 - \vec{\rho}_2|) \tag{19}$$

where C_d is the volume of a d-dimensional sphere and $\vec{\nabla}_1$ is the derivative with respect to $\vec{\rho}_1$. Note that $\bar{u}_0^{(1)}$ on the left hand side of (19) is given by $\bar{u}_0$ (14) whereas that on the right hand side by $\bar{u}_0 = -G\theta$ from (9) and (12) for $\varepsilon \to 0$ and $|\vec{\rho}_1 - \vec{\rho}_2| > R$.

Comparing (19) with (17), one notes that the interaction term is added on the right hand side of (17). Since $\bar{u}_0$ is isotropic around the center of gravity, one finally obtains

$$m\ddot{\vec{\rho}}_1 + c(\tau - \tau_c)\dot{\vec{\rho}}_1 + g|\dot{\vec{\rho}}_1|^2\dot{\vec{\rho}}_1 = C_d\vec{\nabla}_1\bar{u}_0(|\vec{\rho}_1 - \vec{\rho}_2|) \tag{20}$$

where the positive constants m, c, τ_c and g can be evaluated from (17). In fact, we have verified that these coefficients in one dimension agree precisely with the previous results [13].

Discussion

In the limit that two pulses are infinitely apart, the second and the third terms in Eq. (20) are dominant and imply that there is a supercritical bifurcation such that a motionless pulse is stable for $\tau > \tau_c$ and begins to propagate for $\tau < \tau_c$. Just at post-threshold, the velocity of a pulse is arbitrarily small so that a pair of pulses cannot overcome the energy barrier due to the repulsive interaction given by the right hand side of (20). Since there is an inertia term with $\ddot{\rho}_1$ those pulses have to move back after collision. This expectation has indeed been confirmed by solving (20) numerically. Thus Eq. (20) accounts for the reason why a reflection of pulses upon collision occurs only in a restricted parameter region.

Note that the short time expansion (10) and the smallness of the deviation around the circular shape of a pulse can be justified as far as the velocity of a pulse is sufficiently small and that this is the very condition where the reflection occurs. Therefore the theory has a complete internal consistency. Effects of deformation of pulses away from the bifurcation threshold will be studied separately in the near future.

References

1. H. Sevcikova and M. Marek, in *Bifurcation and Chaos*, R. Seydel et. al. (Eds), Birkhauser Verlag, Basel, (1991)
2. V. Petrov et al, *Phil. Trans. R. Soc. Lond.*, **A347**, 631 (1994)
3. K. Krisher and A. Mikhailov, *Phys. Rev. Letters*, **73**, 3165 (1994)
4. S. Kawaguchi and M. Mimura, *SIAM J. Appl. Math.*, **59**, 920 (1999)
5. R. Kobayashi, T. Ohta and Y. Hayase, *Phys. Rev.*, **E50**, R3291 (1994)
6. O. A. Mornev et al, *Doklady Biophysics*, **346-348**, 21 (1996)
7. O. V. Aslanidi and O. A. Mornev, *J. Bio. Phys.*, **25**, 149 (1999)
8. J.E. Pearson, *Science*, **261**, 189 (1993)
9. W.N. Reynolds at al, *Phys. Rev. Letters*, **72**, 2797 (1994)
10. K.J. Lee et al, *Nature*, **369**, 215 (1994)
11. Y. Hayase, *J. Phys. Soc. Jpn.*, **66**, 2584 (1997)
12. Y. Hayase and T. Ohta, *Phys. Rev. Letters*, **81**, 1726 (1998)
13. T. Ohta, J. Kiyose and M. Mimura, *J. Phys. Soc. Jpn.*, **66**, 1551 (1997)